QUASI-BIALGEBRA STRUCTURES AND TORSION-FREE ABELIAN GROUPS

ALESSANDRO ARDIZZONI, DANIEL BULACU, AND CLAUDIA MENINI

Dedicated to Toma Albu and Constantin Năstăsescu

Abstract. We describe all the quasi-bialgebra structures of a group algebra over a torsion-free abelian group. They all come out to be triangular in a unique way. Moreover, up to an isomorphism, these quasi-bialgebra structures produce only one (braided) monoidal structure on the category of their representations. Applying these results to the algebra of Laurent polynomials, we recover two braided monoidal categories introduced in [CG] by S. Caenepeel and I. Goyvaerts in connection with Hom-structures (Lie algebras, algebras, coalgebras, Hopf algebras).

Introduction

Let $\mathcal{C}$ be a category. In [CG, Section 1], S. Caenepeel and I. Goyvaerts introduce the so called Hom-category $\mathcal{H}(\mathcal{C})$ in order to investigate Hom-structures (Lie algebras, algebras, coalgebras, Hopf algebras) from the monoidal categorical point of view. More exactly, if $\mathcal{C}$ is the category of modules over a commutative ring, then $\mathcal{H}(\mathcal{C})$ admits a symmetric monoidal structure with respect to which (co)algebras in $\mathcal{H}(\mathcal{C})$ coincide with Hom-(co)algebras, Hopf algebras with Hom-Hopf algebras and Lie algebras with Hom-Lie algebras, respectively.

Now, fix a field $\Bbbk$ and denote by $\mathfrak{M}$ the category of $\Bbbk$-vector spaces. The category $\mathcal{H}(\mathfrak{M})$ has objects pairs (V, f_V) with $V \in \mathfrak{M}$ and $f_V \in \mathrm{Aut}_{\Bbbk}(V)$. A module over the polynomial ring $\Bbbk[X]$ is a $\Bbbk$-vector space V together with an element $g_V \in \mathrm{End}_{\Bbbk}(M)$. In order to have g_V invertible, as in the case of $\mathcal{H}(\mathfrak{M})$, the ring $\Bbbk[X]$ must be replaced with the algebra of Laurent polynomials $\Bbbk\left[X, X^{-1}\right]$ or, equivalently, with the group algebra $\Bbbk[\mathbb{Z}]$. These facts suggest a connection between the category $\mathcal{H}(\mathfrak{M})$ and the category of $\Bbbk[\mathbb{Z}]$-modules. They are actually isomorphic, and this will be proved in Proposition 3.2.

As a matter of fact, it was proved in [CG] that the category $\mathcal{H}(\mathfrak{M})$ has two different braided monoidal structures, denoted by $\mathcal{H}(\mathfrak{M})$ and $\widetilde{\mathcal{H}}(\mathfrak{M})$, respectively. This leads us to consider braided monoidal structures on the category of $\Bbbk[\mathbb{Z}]$-modules. We restrict ourselves to the case when these structures are induced by the strict monoidal structure of $\mathfrak{M}$. It comes out that in this case we have to compute the quasi-bialgebra structures of the group algebra $\Bbbk[\mathbb{Z}]$, cf. Theorem 2.1. We will do this in a wider context, by replacing $\mathbb{Z}$ with a torsion-free abelian group. In fact, our arguments are valid for any abelian group G with the property that, for any natural number $n \in \{1,2,3\}$, the group of units of the group algebra $\Bbbk[G^n]$ is trivial (that is, any invertible element of $\Bbbk[G^n]$ is a nonzero scalar multiple of an element in G^n), where G^n stands for the direct product of n copies of G. But, if this is the case then by [Pa, Lemma 1.1] we have that G is torsion-free (note that the exceptions listed in the Lemma have $|G| = 2$ and hence they fulfill the requirement only for $n = 1$). For the other way around, if G is a torsion-free abelian group then by [GH, Corollary 2.5], inductively, it follows that $\Bbbk[G^n]$ has the group of units trivial. (Note that this can be obtained also from [Pa, Lemmas 1.6, 1.7 & 1.9(ii)]).

2010 *Mathematics Subject Classification.* Primary 16W30; Secondaries 18D10; 16S34.

Key words and phrases. Quasi-bialgebras, Hom-category, Laurent polynomials, torsion-free abelian groups.

This paper was written while the first and the third authors were members of GNSAGA. The first author was partially supported by the research grant “Progetti di Eccellenza 2011/2012” from the “Fondazione Cassa di Risparmio di Padova e Rovigo”. The second author was supported by the strategic grant POSDRU/89/1.5/S/58852, Project “Postdoctoral program for training scientific researchers” cofinanced by the European Social Fund within the Sectorial Operational Program Human Resources Development 2007 - 2013.

Now, for a torsion-free abelian group we show that the third Harrison cohomology group $H^3_{\mathrm{Harr}}(\Bbbk[G],\Bbbk,\mathbb{G}_m)$ is trivial (Proposition 2.5). Moreover, any Harrison 3-cocycle on $\Bbbk[G]$ is uniquely determined by a pair (h,g) of elements of G, and this allows us to describe, up to an isomorphism, all the quasi-bialgebra structures on the group algebra $\Bbbk[G]$. When we specialize this for the multiplicative cyclic group $\langle g\rangle\cong\mathbb{Z}$ we obtain that the quasi-bialgebra structures on the group algebra $\Bbbk[\langle g\rangle]$ are, up to an isomorphism, completely determined by triples $(q,a,b)\in(\Bbbk\backslash\{0\})\times\mathbb{Z}\times\mathbb{Z}$, see Theorem 2.8. Furthermore, all of them are deformations of the ordinary bialgebra structure of $\Bbbk[\langle g\rangle]$ by an invertible element in $\Bbbk[\langle g\rangle]\otimes\Bbbk[\langle g\rangle]$, and so, up to an isomorphism, the category ${}_{\Bbbk[\langle g\rangle]}\mathfrak{M}$ admits a unique (strict) monoidal structure. The same is valid for the braided situation, and this is mostly because the ordinary bialgebra $\Bbbk[\langle g\rangle]$ has a unique quasi-triangular (actually triangular) structure (Corollary 2.12).

As we have already explained, the categories $\mathcal{H}(\mathfrak{M})$ and ${}_{\Bbbk[\langle g\rangle]}\mathfrak{M}$ are isomorphic. Consequently, we have a one to one correspondence between the (braided) monoidal structures on $\mathcal{H}(\mathfrak{M})$ and the (braided) monoidal structures on ${}_{\Bbbk[\langle g\rangle]}\mathfrak{M}$. In Theorem 3.4 we endow $\mathcal{H}(\mathfrak{M})$ with the symmetric monoidal category structures induced by those of ${}_{\Bbbk[\langle g\rangle]}\mathfrak{M}$ that we previously computed. In general, such a structure depends on a triple $(q,a,b)\in(\Bbbk\backslash\{0\})\times\mathbb{Z}\times\mathbb{Z}$, and this is why we denoted it by $\mathcal{H}^{a,b}_p(\mathfrak{M})$. We have isomorphisms of symmetric monoidal categories, ${}_{\Bbbk[\langle g\rangle]}\mathfrak{M}\cong{}_{\Bbbk[\langle g\rangle]^{a,b}_q}\mathfrak{M}\cong\mathcal{H}^{a,b}_q(\mathfrak{M})$, see Corollary 3.5. Since $\mathcal{H}^{0,0}_1(\mathfrak{M})$ and $\mathcal{H}^{1,-1}_1(\mathfrak{M})$ can be identified, as symmetric monoidal categories, with $\mathcal{H}(\mathfrak{M})$ and $\widetilde{\mathcal{H}}(\mathfrak{M})$, respectively, we obtain that $\mathcal{H}(\mathfrak{M})$ and $\widetilde{\mathcal{H}}(\mathfrak{M})$ are isomorphic as symmetric monoidal categories (Proposition 3.7 and Corollary 3.8). We recover in this way [CG, Proposition 1.7], in the particular case when the base category is $\mathfrak{M}$.

1. Preliminaries

In this section, we shall fix some basic notation and terminology.

Notation 1.1. *Throughout this paper $\Bbbk$ will denote a field. All vector spaces will be defined over $\Bbbk$. The unadorned tensor product $\otimes$ will denote the tensor product over $\Bbbk$ if not stated otherwise. The category of vector spaces will be denoted by $\mathfrak{M}$.*

Monoidal Categories.([Ka, Chap. XI]) Recall that a *monoidal category* is a category $\mathcal{M}$ endowed with an object $\mathbf{1}\in\mathcal{M}$ (called *unit*), a functor $\otimes:\mathcal{M}\times\mathcal{M}\to\mathcal{M}$ (called *tensor product*), and functorial isomorphisms $a_{X,Y,Z}:(X\otimes Y)\otimes Z\to X\otimes(Y\otimes Z)$, $l_X:\mathbf{1}\otimes X\to X$, $r_X:X\otimes\mathbf{1}\to X$, for every X,Y,Z in $\mathcal{M}$. The functorial morphism a is called the *associativity constraint* and satisfies the *Pentagon Axiom,* that is the following relation

$$(U\otimes a_{V,W,X})\circ a_{U,V\otimes W,X}\circ(a_{U,V,W}\otimes X)=a_{U,V,W\otimes X}\circ a_{U\otimes V,W,X}$$

holds true, for every U,V,W,X in $\mathcal{M}$. The morphisms l and r are called the *unit constraints* and they obey the *Triangle Axiom,* that is $(V\otimes l_W)\circ a_{V,\mathbf{1},W}=r_V\otimes W$, for every V,W in $\mathcal{M}$.

A *monoidal functor* (also called strong monoidal in the literature)

$$(F,\phi_0,\phi_2):(\mathcal{M},\otimes,\mathbf{1},a,l,r)\to(\mathcal{M}',\otimes',\mathbf{1}',a',l',r')$$

between two monoidal categories consists of a functor $F:\mathcal{M}\to\mathcal{M}'$, an isomorphism $\phi_2(U,V):F(U)\otimes' F(V)\to F(U\otimes V)$, natural in $U,V\in\mathcal{M}$, and an isomorphism $\phi_0:\mathbf{1}'\to F(\mathbf{1})$ such that the diagram

$$\begin{tikzcd}
(F(U)\otimes' F(V))\otimes' F(W) \arrow[r, "\phi_2(U,V)\otimes' F(W)"] \arrow[d, "a'_{F(U),F(V),F(W)}"'] & F(U\otimes V)\otimes' F(W) \arrow[r, "\phi_2(U\otimes V,W)"] & F((U\otimes V)\otimes W) \arrow[d, "F(a_{U,V,W})"] \\
F(U)\otimes'(F(V)\otimes' F(W)) \arrow[r, "F(U)\otimes'\phi_2(V,W)"] & F(U)\otimes' F(V\otimes W) \arrow[r, "\phi_2(U,V\otimes W)"] & F(U\otimes(V\otimes W))
\end{tikzcd}$$

is commutative, and the following conditions are satisfied:

$$F(l_U)\circ\phi_2(\mathbf{1},U)\circ(\phi_0\otimes' F(U))=l'_{F(U)},\quad F(r_U)\circ\phi_2(U,\mathbf{1})\circ(F(U)\otimes'\phi_0)=r'_{F(U)}.$$

The monoidal functor is called *strict* if the isomorphisms ϕ_0, ϕ_2 are identities of $\mathcal{M}'$. A *braided monoidal category* $(\mathcal{M}, c)$ is a monoidal category $(\mathcal{M}, \otimes, \mathbf{1})$ equipped with a *braiding* c, that is an isomorphism $c_{U,V} : U \otimes V \to V \otimes U$, natural in $U, V \in \mathcal{M}$, satisfying, for all $U, V, W \in \mathcal{M}$,

$$\begin{aligned} a_{V,W,U} \circ c_{U,V\otimes W} \circ a_{U,V,W} &= (V \otimes c_{U,W}) \circ a_{V,U,W} \circ (c_{U,V} \otimes W), \\ a^{-1}_{W,U,V} \circ c_{U\otimes V,W} \circ a^{-1}_{U,V,W} &= (c_{U,W} \otimes V) \circ a^{-1}_{U,W,V} \circ (U \otimes c_{V,W}). \end{aligned}$$

Such a category is called *symmetric* if we further have $c_{V,U} \circ c_{U,V} = \mathrm{Id}_{U\otimes V}$ for every $U, V \in \mathcal{M}$.

A *(symmetric) braided monoidal functor* is a monoidal functor $F : \mathcal{M} \to \mathcal{M}'$ such that

$$F(c_{U,V}) \circ \phi_2(U, V) = \phi_2(V, U) \circ c'_{F(U),F(V)}.$$

More details on these topics can be found in [Ka, Chapter XIII].

Quasi-Bialgebras. The following definition is not the original one given in [Dr, page 1421]. We adopt the more general form of [Dr, Remark 1, page 1423] (see also [Ka, Proposition XV.1.2]) in order to comprise the case of Hom-categories.

Definition 1.2. A *quasi-bialgebra* is a datum $(H, m, u, \Delta, \varepsilon, \phi, \lambda, \rho)$ where

- (H, m, u) is an associative algebra;
- $\Delta : H \to H \otimes H$ and $\varepsilon : H \to \Bbbk$ are algebra maps;
- $\lambda, \rho \in H$ are invertible elements;
- $\phi \in H \otimes H \otimes H$ is a counital 3-cocycle i.e. it is an invertible element and satisfies

$$\begin{aligned} (H \otimes H \otimes \Delta)(\phi) \cdot (\Delta \otimes H \otimes H)(\phi) &= (1_H \otimes \phi) \cdot (H \otimes \Delta \otimes H)(\phi) \cdot (\phi \otimes 1_H), \\ (H \otimes \varepsilon \otimes H)(\phi) &= \rho \otimes \lambda^{-1}; \end{aligned}$$

- Δ is quasi-coassociative and counitary i.e. it satisfies

$$\begin{aligned} (H \otimes \Delta)\Delta(h) &= \phi \cdot (\Delta \otimes H)\Delta(h) \cdot \phi^{-1}, \\ (\varepsilon \otimes H)\Delta(h) &= \lambda^{-1} h \lambda, \\ (H \otimes \varepsilon)\Delta(h) &= \rho^{-1} h \rho. \end{aligned}$$

A morphism of quasi-bialgebras (see [Ka, page 371])

$$\Xi : (H, m, u, \Delta, \varepsilon, \phi, \lambda, \rho) \to (H', m', u', \Delta', \varepsilon', \phi', \lambda', \rho')$$

is an algebra homomorphism $\Xi : (H, m, u) \to (H', m', u')$ such that

$$\begin{aligned} (\Xi \otimes \Xi)\Delta &= \Delta'\Xi, \qquad \varepsilon'\Xi = \varepsilon, \qquad (\Xi \otimes \Xi \otimes \Xi)(\phi) = \phi', \\ \Xi(\lambda) &= \lambda', \qquad \Xi(\rho) = \rho'. \end{aligned}$$

It is an isomorphism of quasi-bialgebras if, in addition, it is invertible.

We will use the following standard notation

$$\phi^1 \otimes \phi^2 \otimes \phi^3 := \phi \text{ (summation understood)}.$$

In the case when ϕ is not trivial (that is, a nonzero scalar multiple of $1_H \otimes 1_H \otimes 1_H$) and $\lambda = \rho = 1_H$ we call H an *ordinary quasi-bialgebra.* If ϕ is trivial and $\lambda = \rho = 1_H$ we then land at the classical concept of bialgebra.

The definition of a quasi-bialgebra is based on the formalism of monoidal categories. More exactly, if H is a $\Bbbk$-algebra and $\Delta : H \to H \otimes H$ and $\varepsilon : H \to \Bbbk$ are two algebra morphisms then the category of left H-representations, ${}_H\mathfrak{M}$, endowed with the tensor product defined by Δ and with unit object $\Bbbk$ considered as left H-module via ε is monoidal if and only if H is a quasi-bialgebra. As we will see later on, the existence of the above two morphisms Δ, ε is not needed, as it is implied by the fact that the monoidal structure on $\mathfrak{M}$ restricts to a monoidal structure on ${}_H\mathfrak{M}$. For further use, at this moment recall only the monoidal structure on ${}_H\mathfrak{M}$ produced by a quasi-bialgebra H.

Let $(H, m, u, \Delta, \varepsilon, \phi, \lambda, \rho)$ be a quasi-bialgebra. It is well-known, see [Ka, page 285 and Proposition XV.1.2], that the category ${}_H\mathfrak{M}$ becomes a monoidal category via the following structure.

Given a left H-module V, we denote by $\mu = \mu_V^l : H \otimes V \to V, \mu(h \otimes v) = hv$, its left H-action. The tensor product of two left H-modules V and W is a module via diagonal action i.e. $h(v \otimes w) = h_1 v \otimes h_2 w$. The unit is $\Bbbk$, which is regarded as a left H-module via the trivial action i.e. $h\kappa = \varepsilon(h)\kappa$, for all $h \in H$ and $\kappa \in \Bbbk$. The associativity and unit constraints are defined, for all $V, W, Z \in {}_H\mathfrak{M}$ and $v \in V, w \in W, z \in Z$, by

$$a_{V,W,Z}((v \otimes w) \otimes z) := \phi^1 v \otimes (\phi^2 w \otimes \phi^3 z),$$
$$l_V(1 \otimes v) := \lambda v \qquad \text{and} \qquad r_V(v \otimes 1) := \rho v.$$

The monoidal category we have just described will be denoted by $({}_H\mathfrak{M}, \otimes, \Bbbk, a, l, r)$.

Let $(H, m, u, \Delta, \varepsilon, \phi, \lambda, \rho)$ be a quasi-bialgebra. Given an invertible element $\alpha \in H \otimes H$, we can construct a new quasi-bialgebra $H_\alpha = (H, m, u, \Delta_\alpha, \varepsilon, \phi_\alpha, \lambda_\alpha, \rho_\alpha)$ where

$$\begin{aligned}
\Delta_\alpha(h) &= \alpha \cdot \Delta(h) \cdot \alpha^{-1}, \\
\phi_\alpha &= (1_H \otimes \alpha) \cdot (H \otimes \Delta)(\alpha) \cdot \phi \cdot (\Delta \otimes H)\left(\alpha^{-1}\right) \cdot \left(\alpha^{-1} \otimes 1_H\right), \\
\lambda_\alpha &= \lambda \cdot (\varepsilon_H \otimes H)\left(\alpha^{-1}\right), \\
\rho_\alpha &= \rho \cdot (H \otimes \varepsilon_H)\left(\alpha^{-1}\right).
\end{aligned}$$

By [Ka, Lemma XV.3.4], for every invertible element $\alpha \in H \otimes H$, the identity functor $\mathrm{Id} : {}_H\mathfrak{M} \to {}_{H_\alpha}\mathfrak{M}$ induces a monoidal category isomorphism $\Gamma(\alpha) = (\mathrm{Id}, \alpha_0, \alpha_2) : {}_H\mathfrak{M} \to {}_{H_\alpha}\mathfrak{M}$, where $\alpha_0 = \mathrm{Id}_\Bbbk$ and $\alpha_2(V, W)(v \otimes w) := \alpha^{-1}(v \otimes w)$. The inverse is $\Gamma\left(\alpha^{-1}\right)$.

Notice also that for a quasi-bialgebra H there is always an invertible element $u \in H \otimes H$ such that H_u is an ordinary quasi-bialgebra, and so ${}_H\mathfrak{M}$ is always monoidal isomorphic to a category for which the unit object and the left and right unit constraints are those of $\mathfrak{M}$ (see [Dr] or the proof of Proposition 2.3 below).

Definition 1.3. We refer to [Ka, Proposition XV.2.2] but with a different terminology (cf. [Dr, page 1439]). A quasi-bialgebra $(H, m, u, \Delta, \varepsilon, \phi, \lambda, \rho)$ is called *quasi-triangular* whenever there exists an invertible element $R \in H \otimes H$ such that, for every $h \in H$, one has

$$(\Delta \otimes H)(R) = \left[\begin{array}{c} \left(\phi^2 \otimes \phi^3 \otimes \phi^1\right)\left(R^1 \otimes 1 \otimes R^2\right)\left(\phi^1 \otimes \phi^3 \otimes \phi^2\right)^{-1} \\ \left(1 \otimes R^1 \otimes R^2\right)\left(\phi^1 \otimes \phi^2 \otimes \phi^3\right) \end{array}\right] \tag{1}$$

$$(H \otimes \Delta)(R) = \left[\begin{array}{c} \left(\phi^3 \otimes \phi^1 \otimes \phi^2\right)^{-1}\left(R^1 \otimes 1 \otimes R^2\right)\left(\phi^2 \otimes \phi^1 \otimes \phi^3\right) \\ \left(R^1 \otimes R^2 \otimes 1\right)\left(\phi^1 \otimes \phi^2 \otimes \phi^3\right)^{-1} \end{array}\right] \tag{2}$$

$$\Delta^{cop}(h) = R\Delta(h)R^{-1} \tag{3}$$

where $\phi := \phi^1 \otimes \phi^2 \otimes \phi^3$, $R = R^1 \otimes R^2$.

A *morphism of quasi-triangular quasi-bialgebras* is a morphism $\Xi : H \to H'$ of quasi-bialgebras such that $(\Xi \otimes \Xi)(R) = R'$.

By [Ka, Proposition XV.2.2], ${}_H\mathfrak{M} = ({}_H\mathfrak{M}, \otimes, \Bbbk, a, l, r)$ is braided if and only if there is an invertible element $R \in H \otimes H$ such that $(H, m, u, \Delta, \varepsilon, \phi, \lambda, \rho, R)$ is quasi-triangular. Note that the braiding is given, for all $X, Y \in {}_H\mathfrak{M}$, by

$$c_{X,Y} : X \otimes Y \to Y \otimes X : x \otimes y \mapsto R^2 y \otimes R^1 x.$$

Moreover ${}_H\mathfrak{M}$ is symmetric if and only if we further assume

$$R^2 \otimes R^1 = R^{-1}. \tag{4}$$

Such a quasi-bialgebra will be called a *triangular quasi-bialgebra.* A morphism of triangular quasi-bialgebras is just a morphism of the underlying quasi-triangular quasi-bialgebras structures.

Given an invertible element $\alpha \in H \otimes H$, if H is (quasi-)triangular then so is H_α with respect to

$$R_\alpha = \left(\alpha^2 \otimes \alpha^1\right) R\alpha^{-1}$$

where $\alpha := \alpha^1 \otimes \alpha^2$. This depends on the fact that the monoidal category isomorphism $\Gamma(\alpha) = (\mathrm{Id}, \alpha_0, \alpha_2) : {}_H\mathfrak{M} \to {}_{H_\alpha}\mathfrak{M}$ induces a (symmetric) braided structure on ${}_{H_\alpha}\mathfrak{M}$. In particular we have

$$c_{H,H}^{H_\alpha} = \alpha_2^{-1}(H, H) \circ F\left(c_{H,H}^H\right) \circ \alpha_2(H, H).$$

By [Ka, Proposition XV.2.2], $c^{H_\alpha}_{H,H}$ is of the form $c^{H_\alpha}_{H,H}(x\otimes y)=R^2_\alpha y\otimes R^1_\alpha x$ for all $x,y\in H$ where

$$\begin{aligned}
R^2_\alpha\otimes R^1_\alpha &= c^{H_\alpha}_{H,H}(1_H\otimes 1_H)=\left(\alpha_2^{-1}(H,H)\circ F\left(c^H_{H,H}\right)\circ\alpha_2(H,H)\right)(1_H\otimes 1_H)\\
&= \left(\alpha_2^{-1}(H,H)\circ F\left(c^H_{H,H}\right)\right)\left(\alpha^{-1}(1_H\otimes 1_H)\right)=\left(\alpha_2^{-1}(H,H)\circ F\left(c^H_{H,H}\right)\right)\left(\alpha^{-1}\right)\\
&= \alpha_2^{-1}(H,H)\left(R^2\left(\alpha^{-1}\right)^2\otimes R^1\left(\alpha^{-1}\right)^1\right)\\
&= \alpha\left(R^2\left(\alpha^{-1}\right)^2\otimes R^1\left(\alpha^{-1}\right)^1\right)=\alpha^1R^2\left(\alpha^{-1}\right)^2\otimes\alpha^2R^1\left(\alpha^{-1}\right)^1
\end{aligned}$$

and hence $R_\alpha=R^1_\alpha\otimes R^2_\alpha=\alpha^2R^1\left(\alpha^{-1}\right)^1\otimes\alpha^1R^2\left(\alpha^{-1}\right)^2=\left(\alpha^2\otimes\alpha^1\right)R\alpha^{-1}$ as claimed above.

2. Quasi-triangular quasi-bialgebra structures on a group algebra of a torsion-free abelian group

Let $\Bbbk$ be a field and G a torsion-free abelian group. We are interested in classifying the (braided) monoidal structures on the category of left modules over the group algebra $\Bbbk[G]$ induced by that of $\mathfrak{M}$. We shall see that this is equivalent to the classification, up to deformation by an invertible element, of (quasi-triangular) quasi-bialgebra structures on the group algebra $\Bbbk[G]$.

We start with the monoidal case. In general, for H a $\Bbbk$-algebra, we say that the monoidal structure on $\mathfrak{M}$ restricts to a monoidal structure on ${}_H\mathfrak{M}$ if

(a) for any two left H-modules X,Y the tensor product $X\otimes Y$ in $\mathfrak{M}$ admits a left H-module structure;
(b) the tensor product in $\mathfrak{M}$ of two left H-module morphisms is a morphism in ${}_H\mathfrak{M}$, and so $\otimes$ induces a functor from ${}_H\mathfrak{M}\times{}_H\mathfrak{M}$ to ${}_H\mathfrak{M}$;
(c) $\Bbbk$, as a trivial $\Bbbk$-module, admits a left H-module structure;
(d) there exist functorial isomorphisms $a=\left(a_{X,Y,Z}:(X\otimes Y)\otimes Z\to X\otimes(Y\otimes Z)\right)_{X,Y,Z\in{}_H\mathfrak{M}}$, $l=\left(l_X:\Bbbk\otimes X\to X\right)_{X\in{}_H\mathfrak{M}}$ and $r=\left(r_X:X\otimes\Bbbk\to X\right)_{X\in{}_H\mathfrak{M}}$ in ${}_H\mathfrak{M}$ such that the Pentagon axiom and the Triangle Axiom are satisfied.

The next result is a slightly improved version of [Ka, Proposition XV.1.2] and can be viewed as a reconstruction type theorem for quasi-bialgebras.

Theorem 2.1. *Let $\Bbbk$ be a field and H a $\Bbbk$-algebra. Then there exists a one to one correspondence between*

- *monoidal structures on ${}_H\mathfrak{M}$ induced by the strict monoidal structure of $\mathfrak{M}$;*
- *quasi-bialgebra structures on H.*

Proof. Assume that the strict monoidal structure on $\mathfrak{M}$ induces a monoidal structure on ${}_H\mathfrak{M}$. In particular, this implies that we have a left H-module structure $\cdot:H\otimes(H\otimes H)\to H\otimes H$ on $H\otimes H$. If we define $\Delta:H\to H\otimes H$ given by $\Delta(h)=h\cdot(1_H\otimes 1_H)$, for all $h\in H$, we then claim that Δ is an algebra map. Indeed, it is clear that $\Delta(1_H)=1_H\otimes 1_H$. To see that Δ is multiplicative we proceed as follows. Let $X\in{}_H\mathfrak{M}$ and fix $x\in X$. Then $\varphi_x:H\ni h\mapsto hx\in X$ is a left H-module morphism. Similarly, for $Y\in{}_H\mathfrak{M}$ and $y\in Y$ define $\varphi_y:H\to Y$, a left H-module morphism. According to (a) and (b) we have $\varphi_x\otimes\varphi_y:H\otimes H\to X\otimes Y$ a left H-linear morphism, hence

$$(\varphi_x\otimes\varphi_y)(h\cdot(h'\otimes h''))=h\cdot(h'x\otimes h''y),$$

for all $h,h',h''\in H$. If we take $h'=h''=1_H$ and denote $\Delta(h)=h_1\otimes h_2$ (summation implicitly understood), we then get that $h\cdot(x\otimes y)=h_1x\otimes h_2y$, for all $h\in H$, and so Δ determines completely the left H-module structure on the tensor product $X\otimes Y$. It follows now easily that Δ is multiplicative, providing that $H\otimes H$ has the usual componentwise algebra structure.

We look now at the condition (c). We show that giving a left H-module structure on $\Bbbk$ is equivalent to giving an algebra map $\varepsilon:H\to\Bbbk$. Indeed, let $\cdot:H\otimes\Bbbk\to\Bbbk$ be a left H-module structure on $\Bbbk$. Since $\cdot$ is $\Bbbk$-linear we have $h\cdot\kappa=h\cdot(\kappa 1_\Bbbk)=\kappa(h\cdot 1_\Bbbk)=(\kappa h)\cdot 1_\Bbbk$, for all $\kappa\in\Bbbk$ and $h\in H$. So if we define $\varepsilon:H\to\Bbbk$ given by $\varepsilon(h):=h\cdot 1_\Bbbk$, for all $h\in H$, then $\kappa\varepsilon(h)=\varepsilon(\kappa h)$, for

all $h \in H$ and $\kappa \in \Bbbk$. Otherwise stated, ε is $\Bbbk$-linear and $h \cdot \kappa = (\kappa h) \cdot 1_{\Bbbk} = \varepsilon(\kappa h) = \kappa\varepsilon(h)$, for all $\kappa \in \Bbbk$ and $h \in H$. Now, ε is multiplicative since, for all $h, g \in H$,

$$\varepsilon(hg) = (hg) \cdot 1_{\Bbbk} = h \cdot (g \cdot 1_{\Bbbk}) = h \cdot \varepsilon(g) = \varepsilon(\varepsilon(g)h) = \varepsilon(h)\varepsilon(g).$$

In addition, $\varepsilon(1_H) = 1_H \cdot 1_{\Bbbk} = 1_{\Bbbk}$, and therefore ε is an algebra morphism.

Conversely, if $\varepsilon : H \to \Bbbk$ is an algebra map then clearly $\Bbbk$ is a left H-module via the structure defined by $h \cdot \kappa = \varepsilon(h)\kappa$, for all $h \in H$ and $\kappa \in \Bbbk$. It is immediate that the two correspondences defined above are inverses of each other. As far as we are concerned, retain that (c) implies the existence of an algebra map $\varepsilon : H \to \Bbbk$ such that $h \cdot \kappa = \varepsilon(h)\kappa$, for all $h \in H$ and $\kappa \in \Bbbk$.

Concluding, the desired monoidal structure on ${}_H\mathfrak{M}$ is induced by a triple (H, Δ, ε) as in the statement of [Ka, Proposition XV.1.2]. Consequently, $(H, \Delta, \varepsilon, \phi, \lambda, \rho)$ is a quasi-bialgebra, where

$$\phi = a_{H,H,H}(1_H \otimes 1_H \otimes 1_H)\ ,\ \lambda = l_H(1_{\Bbbk} \otimes 1_H) \text{ and } \rho = r_H(1_H \otimes 1_{\Bbbk})\ ,$$

respectively. For the other way around see the comments made after Definition 1.2. □

If we restrict ourself to the strict monoidal case we then get the following well-known result.

Corollary 2.2. *Let H be a $\Bbbk$-algebra. Then there is a one to one correspondence between*

- *the strict monoidal structures on ${}_H\mathfrak{M}$ such that the forgetful functor $\mathfrak{U} : {}_H\mathfrak{M} \to \mathfrak{M}$ is strict monoidal;*
- *bialgebra structures on H.*

If H is a quasi-bialgebra then the forgetful functor $\mathfrak{U} : {}_H\mathfrak{M} \to \mathfrak{M}$ is not necessarily monoidal, although the monoidal structure on ${}_H\mathfrak{M}$ is induced by the strict monoidal structure on $\mathfrak{M}$. More exactly, we have the following situation.

Proposition 2.3. *Let $(H, m, u, \Delta, \varepsilon, \phi, \lambda, \rho)$ be a quasi-bialgebra. Then the forgetful functor $\mathfrak{U} : {}_H\mathfrak{M} \to \mathfrak{M}$ is monoidal if and only if there exists an invertible element $\mathfrak{f} \in H \otimes H$ such that $H_{\mathfrak{f}}$ is an ordinary bialgebra.*

Proof. Let us start by noting that, without loss of generality, we can assume $\lambda = \rho = 1_H$. This observation is due to Drinfeld [Dr] and reduces the study of quasi-bialgebras to those of this type.

Indeed, applying $\varepsilon \otimes \varepsilon$ to the both sides of $(H \otimes \varepsilon \otimes H)(\phi) = \rho \otimes \lambda^{-1}$ we get $\varepsilon(\phi^1)\varepsilon(\phi^2)\varepsilon(\phi^3) = \varepsilon(\rho)\varepsilon(\lambda^{-1})$. On the other hand, it follows from $\varepsilon(h_1)h_2 = \lambda^{-1}h\lambda$ that $\varepsilon(h_1)\varepsilon(h_2) = \varepsilon(h)$, for any $h \in H$. Therefore, applying $\varepsilon \otimes \varepsilon \otimes \varepsilon \otimes \varepsilon$ to the both sides of

$$(H \otimes H \otimes \Delta)(\phi)(\Delta \otimes H \otimes H)(\phi) = (1_H \otimes \phi)(H \otimes \Delta \otimes H)(\phi)(\phi \otimes 1_H)$$

and using that $\varepsilon(\phi^1)\varepsilon(\phi^2)\varepsilon(\phi^3)$ is invertible in $\Bbbk$ we obtain $\varepsilon(\phi^1)\varepsilon(\phi^2)\varepsilon(\phi^3) = 1$. Correlated to $\varepsilon(\phi^1)\varepsilon(\phi^2)\varepsilon(\phi^3) = \varepsilon(\rho)\varepsilon(\lambda^{-1})$ this yields $\varepsilon(\lambda) = \varepsilon(\rho)$.

Let us denote $c := \varepsilon(\lambda) = \varepsilon(\rho)$. If $u = c^{-1}\rho \otimes \lambda$ then one can see easily that u is invertible and $\lambda_u = \rho_u = 1_H$. Thus H_u is a quasi-bialgebra for which $\lambda_u = \rho_u = 1_H$. Furthermore, $\Gamma(u) : {}_H\mathfrak{M} \to {}_{H_u}\mathfrak{M}$ is a monoidal category isomorphism and $\mathfrak{U}_u \circ \Gamma(u) = \mathfrak{U}$, where we denoted by $\mathfrak{U}_u : {}_{H_u}\mathfrak{M} \to \mathfrak{M}$ the corresponding forgetful functor. Since a composition of monoidal functors is monoidal as well, we get that $\mathfrak{U}$ is monoidal if and only if $\mathfrak{U}_u$ is so. So we reduced the problem to the case when $\lambda = \rho = 1_H$, as desired.

Assume now that $(\mathfrak{U}, \phi_0, \phi_2) : {}_H\mathfrak{M} \to \mathfrak{M}$ is a monoidal functor, and take $v = \phi_2(H, H)(1_H \otimes 1_H) \in H \otimes H$. We first show that v is invertible, and that it determines completely ϕ_2. To this end, let X, Y be two left H-modules and fix $x \in X$ and $y \in Y$. If $\varphi_x : H \to X$ and $\varphi_y : H \to Y$ are the left H-module morphisms defined in the proof of Theorem 2.1 then by the naturalness of ϕ_2 we obtain

$$(\varphi_x \otimes \varphi_y)\phi_2(H, H) = \phi_2(X, Y)(\varphi_x \otimes \varphi_y).$$

Evaluating the above equality in $1_H \otimes 1_H$ we get $\phi_2(X, Y)(x \otimes y) = v^1x \otimes v^2y$, where $v = v^1 \otimes v^2 \in H \otimes H$.

By similar arguments applied now to ϕ_2^{-1}, the inverse of ϕ_2, we deduce that $w = \phi_2^{-1}(H, H)(1_H \otimes 1_H) \in H \otimes H$ determines completely ϕ_2^{-1}. More exactly, for $X, Y \in {}_H\mathfrak{M}$ we have $\phi_2^{-1}(X, Y)(x \otimes y) =$

$w^1x\otimes w^2y$, for all $x\in X$ and $y\in Y$, where $w=w^1\otimes w^2$. Since ϕ_2 and ϕ_2^{-1} are inverses it follows now that v is invertible with $v^{-1}=w$.

Finally, the commutativity of the diagram involving ϕ_2 in the definition of a monoidal functor comes out as

$$(H\otimes\Delta)(v)\cdot(1_H\otimes v)\cdot(x\otimes y\otimes z)=\phi\cdot(\Delta\otimes H)(v)\cdot(v\otimes 1_H)\cdot(x\otimes y\otimes z),$$

for all $X,Y,Z\in {}_H\mathfrak{M}$ and $x\in X$, $y\in Y$ and $z\in Z$. Clearly this is equivalent to the fact that $\phi_{v^{-1}}=1_H\otimes 1_H\otimes 1_H$.

Likewise, if $\phi_0:\Bbbk\to\Bbbk$ is a $\Bbbk$-linear isomorphism then there is a nonzero $\varsigma\in\Bbbk$ such that $\phi_0(\kappa)=\varsigma\kappa$, for all $\kappa\in\Bbbk$. Consequently, the required equations for ϕ_0 in the definition of a monoidal functor are equivalent to $\varsigma\varepsilon(v^1)v^2=1_H=\varsigma\varepsilon(v^2)v^1$, and so $\lambda_{\varsigma^{-1}v^{-1}}=\rho_{\varsigma^{-1}v^{-1}}=1_H$. Since $\phi_{\varsigma^{-1}v^{-1}}=\phi_{v^{-1}}=1_H\otimes 1_H\otimes 1_H$ we conclude that for the invertible element $\mathfrak{f}=\varsigma^{-1}v^{-1}$ of $H\otimes H$ the quasi-bialgebra $H_{\mathfrak{f}}$ is actually an ordinary bialgebra, as needed.

The converse is immediate since if $\mathfrak{f}\in H\otimes H$ is invertible such that $H_{\mathfrak{f}}$ is an ordinary bialgebra then, according to Corollary 2.2, the forgetful functor $\mathfrak{U}_{\mathfrak{f}}:{}_{H_{\mathfrak{f}}}\mathfrak{M}\to\mathfrak{M}$ is monoidal. Corroborated to the fact that $\Gamma(f):{}_H\mathfrak{M}\to{}_{H_{\mathfrak{f}}}\mathfrak{M}$ is a monoidal category isomorphism such that $\mathfrak{U}_{\mathfrak{f}}\circ\Gamma(\mathfrak{f})=\mathfrak{U}$ this leads us to the desired conclusion. So our proof is finished. $\square$

We shall apply the above results to the group algebra $\Bbbk[G]$. By Theorem 2.1, the monoidal structures on ${}_{\Bbbk[G]}\mathfrak{M}$ induced by that of $\mathfrak{M}$ are given by the quasi-bialgebra structures on $\Bbbk[G]$. We next see that these structures are built on the ordinary bialgebra structure of the group algebra $\Bbbk[G]$, providing that G is torsion-free and abelian.

Lemma 2.4. *Let G be a torsion-free abelian group and $\Bbbk[G]$ the group algebra over the field $\Bbbk$ associated to it, and endowed with the ordinary bialgebra structure, that is, endowed with the coalgebra structure given by*

$$\Delta(g)=g\otimes g,\ \varepsilon(g)=1,$$

for all $g\in G$, extended by linearity and as algebra morphisms. Suppose that the group algebra $\Bbbk[G]$ admits a quasi-bialgebra structure given by the comultplication $\widetilde{\Delta}$, counit $\widetilde{\varepsilon}$ and elements ϕ, λ and ρ. Then $(\Bbbk[G],\widetilde{\Delta},\widetilde{\varepsilon})$ is a bialgebra isomorphic to the ordinary bialgebra structure of $\Bbbk[G]$.

Proof. First note that, by [GH, Corollary 2.5], if G is a torsion-free abelian group then the invertible elements in $\Bbbk\left[G\right]$ are exactly those of the form qh where $q\in\Bbbk\backslash\{0\}$ and $h\in G$. Now, since

$$\Bbbk[G]\otimes\Bbbk[G]\ni h\otimes g\mapsto(h,g)\in\Bbbk[G\times G],$$

extended by linearity, defines a bialgebra isomorphism and $G\times G$ is a torsion-free abelian group as well, we deduce that the invertible elements in $\Bbbk[G]\otimes\Bbbk[G]$ are of the form $qh\otimes g$ with $q\in\Bbbk\backslash\{0\}$ and $h,g\in G$.

Suppose now that the group algebra $\Bbbk[G]$ admits a quasi-bialgebra structure given by the comultplication $\widetilde{\Delta}$, counit $\widetilde{\varepsilon}$ and elements ϕ, λ and ρ. Denote $\Bbbk[G]$ with this quasi-bialgebra structure by $\widetilde{\Bbbk[G]}$. Since $\widetilde{\Delta}$ is an algebra map, we get that $\widetilde{\Delta}\left(g\right)$ is invertible in $\widetilde{\Bbbk[G]}\otimes\widetilde{\Bbbk[G]}$, so we can write $\widetilde{\Delta}\left(g\right)=q_gx\otimes y$ for some $q_g\in\Bbbk\backslash\{0\}$ and $x,y\in G$. From

$$\left(\widetilde{\varepsilon}\otimes\widetilde{\Bbbk[G]}\right)\widetilde{\Delta}\left(h\right)=\lambda^{-1}h\lambda,\qquad\left(\widetilde{\Bbbk[G]}\otimes\widetilde{\varepsilon}\right)\widetilde{\Delta}\left(h\right)=\rho^{-1}h\rho,\ \text{for every } h\in\widetilde{\Bbbk[G]}$$

and $\widetilde{\Bbbk[G]}$ commutative, we have that

$$\left(\widetilde{\varepsilon}\otimes\widetilde{\Bbbk[G]}\right)\widetilde{\Delta}\left(h\right)=h,\qquad\left(\widetilde{\Bbbk[G]}\otimes\widetilde{\varepsilon}\right)\widetilde{\Delta}\left(h\right)=h,\ \text{for every } h\in\widetilde{\Bbbk[G]}.$$

Thus Δ is counital and we have $g=q_g\widetilde{\varepsilon}\left(x\right)y$, and hence $y=q_g^{-1}\widetilde{\varepsilon}(x^{-1})g$. Similarly, from $g=q_g\widetilde{\varepsilon}\left(y\right)x$ we get $x=q_g^{-1}\widetilde{\varepsilon}(y^{-1})g$. Summing up we deduce that $\widetilde{\Delta}\left(g\right)=(q_g\widetilde{\varepsilon}(x)\widetilde{\varepsilon}(y))^{-1}g\otimes g=\widetilde{\varepsilon}(g^{-1})g\otimes g$, for all $g\in G$. It is clear now that $\widetilde{\Bbbk[G]}$ has actually an ordinary bialgebra structure, and that $\widetilde{\Bbbk[G]}\ni g\mapsto\widetilde{\varepsilon}(g)g\in\Bbbk[G]$, extended by linearity, is a bialgebra isomorphism. So we are done. $\square$

So if G is a torsion-free abelian group then, up to an isomorphism, the quasi-bialgebra structures on the group algebra $\Bbbk[G]$ are built on the ordinary bialgebra structure (only ϕ, λ, ρ can be non-trivial) of $\Bbbk[G]$, by considering it as a quasi-bialgebra via a so called Harrison 3-cocycle, see for instance [BCT]. Thus our problem reduces to the computation of $H^3_{\mathrm{Harr}}(\Bbbk[G], \Bbbk, \mathbb{G}_m)$. In the sequel we will prove that this cohomology group is trivial. Actually, we will prove that $H^n_{\mathrm{Harr}}(\Bbbk[G], \Bbbk, \mathbb{G}_m)$ is trivial for all $n \geq 2$ and G a torsion free-abelian group.

First, we recall from [Ca, & 9.2] the definition of the Harrison cohomology over a commutative bialgebra over a field.

Let H be a commutative $\Bbbk$-bialgebra and for $n \in \mathbb{N}$ denote by $H^{\otimes n}$ the tensor product over $\Bbbk$ of n copies of H. By convention $H^{\otimes 0} = \Bbbk$. For a fixed $n \in \mathbb{N}$ define the maps $f_0, \cdots, f_{n+1} : H^{\otimes n} \to H^{\otimes n+1}$ given, for all $h_1, \cdots, h_n \in H$, by

$$\begin{aligned}
&f_0(h_1 \otimes \cdots \otimes h_n) = 1_H \otimes h_1 \otimes \cdots \otimes h_n,\\
&f_i(h_1 \otimes \cdots \otimes h_i) = h_1 \otimes \cdots h_{i-1} \otimes \Delta(h_i) \otimes h_{i+1} \otimes \cdots \otimes h_n, \text{ for } i = 1, \cdots, n,\\
&f_{n+1}(h_1 \otimes \cdots \otimes h_n) = h_1 \otimes \cdots \otimes h_n \otimes 1_H.
\end{aligned}$$

Let $\mathbb{G}_m$ be the functor from the category of commutative $\Bbbk$-algebras to the category of abelian groups that maps a commutative $\Bbbk$-algebra A to its group of units, $\mathbb{G}_m(A)$. At the level of morphisms $\mathbb{G}_m$ sends an algebra map $f : A \to B$ to its restriction and corestriction at $\mathbb{G}_m(A)$ and $\mathbb{G}_m(B)$, respectively.

If we set $\delta_n = \prod_{i=0}^{n+1} \mathbb{G}_m(f_i)^{(-1)^i}$, for all $n \in \mathbb{N}$, then we get a complex

$$1 \to \mathbb{G}_m(\Bbbk) \xrightarrow{\delta_0} \mathbb{G}_m(H) \xrightarrow{\delta_1} \mathbb{G}_m(H^{\otimes 2}) \xrightarrow{\delta_2} \cdots$$

The cohomology groups associated to this complex are denoted by $H^n_{\mathrm{Harr}}(H, \Bbbk, \mathbb{G}_m)$, $n \geq 1$. $H^n_{\mathrm{Harr}}(H, \Bbbk, \mathbb{G}_m)$ is called the nth Harrison cohomology group of H with values in $\mathbb{G}_m$.

Proposition 2.5. *Let G be a torsion-free abelian group and $\Bbbk[G]$ the group algebra of G, endowed with the ordinary bialgebra structure. Then $H^0_{\mathrm{Harr}}(\Bbbk[G], \Bbbk, \mathbb{G}_m) = \Bbbk\backslash\{0\}$, $H^1_{\mathrm{Harr}}(\Bbbk[G], \Bbbk, \mathbb{G}_m) = G$ and $H^n_{\mathrm{Harr}}(\Bbbk[G], \Bbbk, \mathbb{G}_m) = 1$, for all $n \geq 2$.*

Proof. As in the proof of Lemma 2.4, inductively, we obtain that the units of $\Bbbk[G]^{\otimes n}$ are of the form $qx_1 \otimes \cdots \otimes x_n$, for a certain $q \in \Bbbk\backslash\{0\}$ and $x_1, \cdots, x_n \in G$.

Therefore, the complex that defines the Harrison cohomology of $\Bbbk[G]$ with coefficients in $\mathbb{G}_m$ has as spaces $\mathbb{G}_m(\Bbbk) = \Bbbk\backslash\{0\}$ and $\mathbb{G}_m(\Bbbk[G]^{\otimes n}) = (\Bbbk\backslash\{0\})\mathbb{G}^{\otimes n}$, $n \geq 1$, and boundary morphisms given by

$$\begin{aligned}
&\delta_0(q) = 1,\ \delta_1(qh) = q1 \otimes 1,\ \delta_2(qh \otimes g) = h^{-1} \otimes 1 \otimes g,\\
&\delta_{2n}(qx_1 \otimes x_2 \otimes \cdots \otimes x_{2n}) = x_1^{-1} \otimes 1 \otimes x_2x_3^{-1} \otimes 1 \otimes \cdots \otimes x_{2n-2}x_{2n-1}^{-1} \otimes 1 \otimes x_{2n}, \text{ for } n \geq 2,\\
&\delta_{2n+1}(qx_1 \otimes x_2 \otimes \cdots \otimes x_{2n+1}) = q1 \otimes x_2 \otimes x_2 \otimes x_4 \otimes x_4 \otimes \cdots \otimes x_{2n} \otimes x_{2n} \otimes 1, \text{ for } n \geq 1,
\end{aligned}$$

for all $q \in \Bbbk\backslash\{0\}$ and $h, g, x_1, \cdots, x_{2n+1} \in G$. We leave the verification of these details to the reader. Notice only that we considered G written multiplicatively and denoted by 1 its neutral element.

Hence, the kernels and the images of the morphisms δ_i, $i \geq 0$, are

$$\mathrm{Ker}(\delta_0) = \Bbbk\backslash\{0\},\ \mathrm{Im}(\delta_0) = 1,\ \mathrm{Ker}(\delta_1) = G,\ \mathrm{Ker}(\delta_2) = \mathrm{Im}(\delta_1) = \Bbbk 1 \otimes 1,$$

and, for $n \geq 1$,

$$\begin{aligned}
&\mathrm{Ker}(\delta_{2n}) = \mathrm{Im}(\delta_{2n-1}) = \{q1 \otimes x_2 \otimes x_2 \otimes \cdots \otimes x_{2n-2} \otimes x_{2n-2} \otimes 1 \mid q \in \Bbbk\backslash\{0\}, x_{2i} \in G\},\\
&\mathrm{Ker}(\delta_{2n+1}) = \mathrm{Im}(\delta_{2n}) = \{x_1 \otimes 1 \otimes x_3 \otimes \cdots \otimes 1 \otimes x_{2n+1} \mid x_{2i+1} \in G\}.
\end{aligned}$$

From here we conclude that $H^n_{\mathrm{Harr}}(\Bbbk[G], \Bbbk, \mathbb{G}_m) = \mathrm{Ker}(\delta_n)/\mathrm{Im}(\delta_{n-1}) = \begin{cases} \Bbbk\backslash\{0\} & \text{if } n = 0 \\ G & \text{if } n = 1 \\ 1 & \text{if } n \geq 2 \end{cases}$, as claimed. □

The computations made in the proof of Proposition 2.5 allow us to determine all the quasi-bialgebra structures on a group algebra of a torsion-free abelian group.

Corollary 2.6. *Let $\Bbbk$ be a field and G a torsion-free abelian group. Then, up to an isomorphism, to give a quasi-bialgebra structure on the group algebra $\Bbbk[G]$ is equivalent to give an element of $(\Bbbk\backslash\{0\})\times G\times G$. More exactly, a quasi-bialgebra structure on $\Bbbk[G]$ is given by the ordinary bialgebra structure of $\Bbbk[G]$,*

$$\phi^{h,g} = h\otimes 1\otimes g,\ \ \lambda = qg^{-1} \text{ and } \rho = qh, \tag{5}$$

for a certain triple $(q,h,g)\in(\Bbbk\backslash\{0\})\times G\times G$. Furthermore, the only ordinary quasi-bialgebra structure that can be built on the ordinary bialgebra structure of $\Bbbk[G]$ is the trivial one, in the sense that it coincides with the ordinary bialgebra structure of $\Bbbk[G]$.

Proof. By the comments made after Lemma 2.4 we have that, up to an isomorphism, the quasi-bialgebra structures built on the group algebra $\Bbbk[G]$ are in a one to one correspondence with the Harrison 3-cocycles on the ordinary bialgebra $\Bbbk[G]$, with coefficients in $\mathbb{G}_m$. Since an element of $\mathrm{Ker}(\delta_3)$ is of the form $h\otimes 1\otimes g$, for some $h,g\in G$, we deduce that the desired quasi-bialgebra structures are completely determined by $\phi^{h,g} := h\otimes 1\otimes g$ and $\lambda,\rho\in(\Bbbk\backslash\{0\})G$ such that $(H\otimes\varepsilon\otimes H)(\phi^{h,g}) = \rho\otimes\lambda^{-1}$. The latest condition is clearly equivalent to the existence of a nonzero scalar q such that $\rho = qh$ and $\lambda = qg^{-1}$. The converse is obvious: for any triple $(q,h,g)\in(\Bbbk\backslash\{0\})\times G\times G$ and $\phi^{h,g}$, λ and ρ as in (5) we have that $(\Bbbk[G],\phi^{h,g},\lambda,\rho)$ is a quasi-bialgebra.

Now, the ordinary quasi-bialgebra structures built on the algebra structure of $\Bbbk[G]$ are those for which $\rho=\lambda=1$. This forces $q=1$ and $h=g=1$, and therefore we land at the ordinary bialgebra structure of $\Bbbk[G]$. □

Observe that for $\Bbbk[G]_q^{h,g} := (\Bbbk[G],\phi^{h,g},\lambda,\rho)$ as in (5) the invertible element u that deforms this quasi-bialgebra structure in an ordinary one is $u = qh\otimes g^{-1}$. A simple inspection shows that $(\Bbbk[G]_q^{h,g})_u = \Bbbk[G]$, and so $\Bbbk[G]_q^{h,g} = \Bbbk[G]_{u^{-1}}$ is a deformation by an invertible element of the ordinary bialgebra structure of $\Bbbk[G]$. In particular this implies the following result.

Corollary 2.7. *Let G be a torsion-free abelian group and $\Bbbk[G]$ the group algebra over $\Bbbk$ associated to G. Then any monoidal structure on the category ${}_{\Bbbk[G]}\mathfrak{M}$ induced by that of $\mathfrak{M}$ is monoidal isomorphic to the strict monoidal category of left representations over the ordinary bialgebra $\Bbbk[G]$.*

Proof. By Theorem 2.1, any monoidal structure on the category ${}_{\Bbbk[G]}\mathfrak{M}$ induced by that of $\mathfrak{M}$ is determined by a quasi-bialgebra structure on $\Bbbk[G]$. By Corollary 2.6, up to isomorphism, these structures are of the form $\Bbbk[G]_q^{h,g}$ as above. Since $(\Bbbk[G]_q^{h,g})_u = \Bbbk[G]$ for $u = qh\otimes g^{-1}$, we get a monoidal category isomorphism $\Gamma(u) : {}_{\Bbbk[G]_q^{h,g}}\mathfrak{M}\to {}_{\Bbbk[G]}\mathfrak{M}$. □

A first example of torsion-free abelian group is $\mathbb{Z}$, the group of integers. In the following we will adopt the multiplicative notation $\langle g\rangle$ for the group $\mathbb{Z}$, where g is a generator.

Theorem 2.8. *Let $(\Bbbk[\langle g\rangle],\Delta,\varepsilon,\phi,\lambda,\rho)$ be a quasi-bialgebra structure on the group algebra $\Bbbk[\langle g\rangle]$. Then, up to an isomorphism, the quasi-bialgebra structure of $\Bbbk[\langle g\rangle]$ is completely determined by some fixed elements $q\in\Bbbk\backslash\{0\}$ and $a,b\in\mathbb{Z}$, in the sense that*

$$\begin{aligned}\Delta(g) &= g\otimes g, \qquad \varepsilon(g)=1,\\ \phi &= g^a\otimes 1_H\otimes g^b,\\ \lambda &= qg^{-b}, \qquad \rho = qg^a.\end{aligned}$$

Furthermore, if we denote this quasi-bialgebra structure on $\Bbbk[\langle g\rangle]$ by $\Bbbk[\langle g\rangle]_q^{a,b}$ then $\Bbbk[\langle g\rangle]_q^{a,b} = \Bbbk[\langle g\rangle]_{q^{-1}g^{-a}\otimes g^b}$. Consequently, up to a monoidal category isomorphism, there is only one monoidal structure on the category of left representations over the group algebra $\Bbbk[\langle g\rangle]$ that is induced by the strict monoidal structure of $\mathfrak{M}$. Namely, the one corresponding to the ordinary bialgebra structure of $\Bbbk[\langle g\rangle]$.

Proof. Follows from Corollaries 2.6 and 2.7, specialized for the case $G=\mathbb{Z}$. □

We move now to the quasi-triangular case. We prove that there is exactly one braided monoidal structure (actually symmetric) on the category of representation of a group algebra associated to a torsion-free abelian group.

PROPOSITION 2.9. *Let G be a torsion-free abelian group, $q \in \Bbbk\backslash\{0\}$ and $h, g \in G$. If $\Bbbk[G]_q^{h,g}$ is the group algebra $\Bbbk[G]$ equipped with the quasi-bialgebra structure from Corollary 2.6 then $R^{h,g} = gh \otimes (gh)^{-1}$ is the only matrix that makes $\Bbbk[G]_q^{h,g}$ a quasi-triangular (actually triangular) quasi-bialgebra. Moreover, $(\Bbbk[G]_q^{h,g}, R^{h,g}) = (\Bbbk[G], 1 \otimes 1)_{q^{-1}h^{-1}\otimes g}$, as triangular quasi-bialgebras.*

Proof. If $u = qh \otimes g^{-1}$ then we have seen that $\Bbbk[G]_q^{h,g} = \Bbbk[G]_{u^{-1}}$. Thus, if $\widetilde{R} \in \Bbbk[G]_q^{h,g} \otimes \Bbbk[G]_q^{h,g}$ endows $\Bbbk[G]_q^{h,g}$ with a quasi-triangular structure then $\widetilde{R}_u$ is an R-matrix for $(\Bbbk[G]_q^{h,g})_u = \Bbbk[G]$. Likewise, if R is an R-matrix on $\Bbbk[G]$ then $R_{u^{-1}}$ defines a quasi-triangular structure on $\Bbbk[G]_{u^{-1}} = \Bbbk[G]_q^{h,g}$. So we have to compute the quasi-triangular structures of the ordinary bialgbra $\Bbbk[G]$.

The definition of a quasi-triangular bialgebra can be obtained from that of a quasi-bialgebra by considering $\phi = 1 \otimes 1 \otimes 1$. So we are looking for an invertible element $R \in \Bbbk[G] \otimes \Bbbk[G]$ such that (3) holds and

$$(\Delta \otimes \Bbbk[G])(R) = (R^1 \otimes 1 \otimes R^2)(1 \otimes R^1 \otimes R^2)\ ,\ (\Bbbk[G] \otimes \Delta)(R) = (R^1 \otimes 1 \otimes R^2)(R^1 \otimes R^2 \otimes 1).$$

Note that, since $\Bbbk[G]$ is both commutative and cocommutative, equation (3) is always true. Since R is invertible, it is of the form $R = tx \otimes y$ for some $t \in \Bbbk\backslash\{0\}$ and $x, y \in G$. So we have

$$tx \otimes x \otimes y = t^2 x \otimes x \otimes y^2 \text{ and } tx \otimes y \otimes y = t^2 x^2 \otimes y \otimes y.$$

The above equalities imply $R = 1 \otimes 1$, thus the bialgebra $\Bbbk[G]$ admits only one R-matrix, the trivial one. From the above we get that $\Bbbk[G]_q^{h,g}$ has a unique quasi-triangular (actually triangular) structure given by

$$R^{h,g} := R_{u^{-1}} = (q^{-1}g \otimes h^{-1})(qh \otimes g^{-1}) = gh \otimes (gh)^{-1}.$$

It is clear that $(\Bbbk[G]_q^{h,g}, R^{h,g}) = (\Bbbk[G], 1 \otimes 1)_{q^{-1}h^{-1}\otimes g}$, as triangular quasi-bialgebras, and this finishes the proof. $\square$

NOTATION 2.10. *Consider the quasi-bialgebra $\Bbbk[G]_q^{h,g}$. In view of Proposition 2.9, there is a unique element R, namely $R^{h,g} = gh \otimes (gh)^{-1}$, such that $\left(\Bbbk[G]_q^{h,g}, R\right)$ is a quasi-triangular (in fact triangular) quasi-bialgebra. By abuse of notation, the datum $\left(\Bbbk[G]_q^{h,g}, R^{h,g}\right)$ will be simply denoted by $\Bbbk[G]_q^{h,g}$.*

From the braided monoidal categorical point of view, up to isomorphism, $\Bbbk[G]_q^{h,g}$ is the "unique" (quasi)triangular quasi-bialgebra structure that can be built on the group algebra $\Bbbk[G]$, in the case when G is a torsion-free abelian group.

COROLLARY 2.11. *Let G be a torsion-free abelian group. Then, up to a braided monoidal category isomorphism, we have a unique braided monoidal structure (actually symmetric) on the category of representations over the group algebra $\Bbbk[G]$, considered monoidal via a structure induced by that of $\mathfrak{M}$. Namely, the one induced by the trivial (quasi)triangular structure of the ordinary bialgebra $\Bbbk[G]$.*

If we take $G = \mathbb{Z}$ and keep the notations as in the statement of Theorem 2.8 we then get the following.

COROLLARY 2.12. *Up to isomorphism, to give a (quasi)triangular quasi-bialgebra structures on the group algebra $\Bbbk[\langle g\rangle]$ is equivalent to give a triple $(q, a, b) \in (\Bbbk\backslash\{0\}) \times \mathbb{Z} \times \mathbb{Z}$. For such a triple (q, a, b) we have a unique (quasi)triangular quasi-bialgebra structure on the group algebra $\Bbbk[\langle g\rangle]$, and this is $\Bbbk[\langle g\rangle]_q^{a,b}$ equipped with the R-matrix $R^{a,b} := g^{a+b} \otimes g^{-a-b}$. Furthermore, $(\Bbbk[\langle g\rangle]_q^{a,b}, R^{a,b}) = (\Bbbk[\langle g\rangle], 1\otimes 1)_{q^{-1}g^{-a}\otimes g^b}$, and so, up to a braided monoidal category isomorphism, the category ${}_{\Bbbk[\langle g\rangle]}\mathfrak{M}$ admits a unique braided monoidal (actually symmetric) structure, if it is considered monoidal via a structure induced by that of $\mathfrak{M}$. Namely, the one obtained from the trivial (quasi)triangular structure of the ordinary bialgebra $\Bbbk[\langle g\rangle]$.*

3. The Hom-category

Let G be a free abelian group. It can be shown that a representation of the group algebra $\Bbbk[G]$ identifies with a pair $(M,(f_g)_{g\in S})$, where M is a $\Bbbk$-vector space and $(f_g)_{g\in S}$ is a family of commuting $\Bbbk$-automorphisms of M indexed by a set of generators of G. This gives us a new description of the category ${}_{\Bbbk[G]}\mathfrak{M}$. In the case when $G=\langle g\rangle$ is the infinite cyclic group we will see that this description of ${}_{\Bbbk[\langle g\rangle]}\mathfrak{M}$ coincides with a so called Hom-category, previously introduced in [CG, Section 1]. This will allow us to describe, up to an isomorphism, all the braided monoidal structure on the Hom-category of $\mathfrak{M}$.

Definition 3.1. Let $\mathcal{C}$ be an ordinary category. We associate to $\mathcal{C}$ a new category $\mathcal{H}(\mathcal{C})$ as follows. Objects are pairs (M,f_M) with $M\in\mathcal{C}$ and $f_M\in \mathrm{Aut}_{\mathcal{C}}(M)$. A morphism $\xi:(M,f_M)\to(N,f_N)$ is a morphism $\xi:M\to N$ in $\mathcal{C}$ such that

$$f_N\circ\xi=\xi\circ f_M. \tag{6}$$

The category $\mathcal{H}(\mathcal{C})$ is called the *Hom-category* associated to $\mathcal{C}$.

In the case when $\mathcal{C}=\mathfrak{M}$ we have the following description for $\mathcal{H}(\mathcal{C})$.

Proposition 3.2. *We have a category isomorphism $W:{}_{\Bbbk[\langle g\rangle]}\mathfrak{M}\to\mathcal{H}(\mathfrak{M})$, given on objects by*

$$W\left(X,\mu_X:\Bbbk[\langle g\rangle]\otimes X\to X\right)=\left(X,f_X:X\to X\right),$$

where $f_X(x):=\mu_X(g\otimes x)$, for all $x\in X$, and on morphisms by $W\xi=\xi$.

Proof. It can be easily seen that

$$\mathrm{Alg}_{\Bbbk}\left(\Bbbk[\langle g\rangle],\mathrm{End}_{\Bbbk}(V)\right)\cong \mathrm{Grp}\left(\langle g\rangle,\mathrm{Aut}_{\Bbbk}(V)\right)\cong\mathrm{Aut}_{\Bbbk}(V).$$

Therefore, to a left $\Bbbk[\langle g\rangle]$-module (V,μ_V) corresponds a pair (V,f_V), where V is a $\Bbbk$-vector space and $f_V\in\mathrm{Aut}_{\Bbbk}(V)$. The correspondence is given by

$$gv:=f_V(v),\ \text{for all } v\in V.$$

A morphism $\xi:(V,\mu_V)\to(W,\mu_W)$ of left $\Bbbk[\langle g\rangle]$-modules corresponds to a $\Bbbk$-linear map $\xi:V\to W$ such that, for every $v\in V,\xi(gv)=g\xi(v)$, i.e., for every $v\in V$, $\xi(f_V(v))=f_W\xi(v)$ or, equivalently, $\xi\circ f_V=f_W\circ\xi$. □

Theorem 3.3. *Let $(\mathcal{A},\otimes,\mathbf{1},a,r,l)$ be a monoidal category, let $\mathcal{A}'$ be a category and let $W:\mathcal{A}\to\mathcal{A}'$ be a category isomorphism. For every $X',Y',Z'\in\mathcal{A}'$ we set $X:=W^{-1}(X'),Y:=W^{-1}(Y')$ and $Z:=W^{-1}(Z')$, and define*

$$\begin{aligned}
X'\otimes' Y' &:= W(X\otimes Y),\qquad \mathbf{1}':=W(\mathbf{1}),\\
l'_{X'} &:= \left(\mathbf{1}'\otimes' X'=W(\mathbf{1}\otimes X)\xrightarrow{Wl_X}WX=X'\right),\\
r'_{X'} &:= \left(X'\otimes'\mathbf{1}'=W(X\otimes\mathbf{1})\xrightarrow{Wr_X}WX=X'\right),\\
a'_{X',Y',Z'} &:= \left((X'\otimes' Y')\otimes' Z'=W((X\otimes Y)\otimes Z)\xrightarrow{Wa_{X,Y,Z}}W(X\otimes(Y\otimes Z))=X'\otimes'(Y'\otimes' Z')\right).
\end{aligned}$$

Then $(\mathcal{A}',\otimes',\mathbf{1}',a',r',l')$ is monoidal. Moreover, $(W,w_0,w_2):(\mathcal{A},\otimes,\mathbf{1},a,r,l)\to(\mathcal{A}',\otimes',\mathbf{1}',a',r',l')$ is a strict monoidal isomorphism functor.

Furthermore, if $(\mathcal{A},\otimes,\mathbf{1},a,r,l,c)$ is (symmetric) braided then so is $(\mathcal{A}',\otimes',\mathbf{1}',a',r',l',c')$, where

$$c'_{X',Y'}=\left(X'\otimes Y'=W(X\otimes Y)\xrightarrow{Wc_{X,Y}}W(Y\otimes X)=Y'\otimes X'\right),$$

and via these structures (W,w_0,w_2) becomes a strict isomorphism of (symmetric) braided monoidal categories.

Proof. It is straightforward, cf. [SR, 4.4.3 and 4.4.5]. □

Theorem 3.4. *Up to isomorphism, the monoidal structures on the category $\mathcal{H}(\mathfrak{M})$ induced by the strict monoidal structure of $\mathfrak{M}$ are completely determined by triples $(q,a,b) \in (\Bbbk\backslash\{0\}) \times \mathbb{Z} \times \mathbb{Z}$. Explicitly, if $(\mathcal{H}(\mathfrak{M}), \otimes, (\Bbbk, f_\Bbbk), l, r)$ is such a structure then there exists $(q,a,b) \in (\Bbbk\backslash\{0\}) \times \mathbb{Z} \times \mathbb{Z}$ such that $\mathcal{H}(\mathfrak{M}) = \mathcal{H}_q^{a,b}(\mathfrak{M})$ as monoidal category, where by $\mathcal{H}_q^{a,b}(\mathfrak{M})$ we denote the category $\mathcal{H}(\mathfrak{M})$ equipped with monoidal structure given by*

$$\begin{aligned}
&(X,f_X) \otimes (Y,f_Y) = (X \otimes Y, f_X \otimes f_Y) \qquad , \qquad (\Bbbk, \mathrm{Id}_\Bbbk),\\
&a_{(X,f_X),(Y,f_Y),(Z,f_Z)}\left((x\otimes y)\otimes z\right) = f_X^a(x) \otimes \left(y \otimes f_Z^b(z)\right), \text{ for all } x\in X,\ y \in Y,\ z\in Z,\\
&l_{(X,f_X)} = l_X : (\Bbbk \otimes X, \mathrm{Id}_\Bbbk \otimes f_X) \to (X,f_X),\ l_X(\kappa \otimes x) = \kappa q f_X^{-b}(x), \text{ for all } \kappa \in \Bbbk,\ x \in X,\\
&r_{(X,f_X)} = r_X : (X \otimes \Bbbk, f_X \otimes \mathrm{Id}_\Bbbk) \to (X,f_X),\ r_X(x \otimes \kappa) = \kappa q f_X^{a}(x), \text{ for all } \kappa \in \Bbbk,\ x \in X.
\end{aligned}$$

Moreover, the monoidal category $\mathcal{H}_q^{a,b}(\mathfrak{M})$ admits a unique braided (actually symmetric) monoidal structure, given by the braiding

$$c_{(X,f_X),(Y,f_Y)} = c_{X,Y} : (X\otimes Y, f_X \otimes f_Y) \to (Y \otimes X, f_Y \otimes f_X),\ c_{X,Y}(x\otimes y) = f_Y^{-a-b}(y) \otimes f_X^{a+b}(x),$$

for all $x \in X$ and $y \in Y$. Consequently, the functor W defined in Proposition 3.2 produces a strict symmetric monoidal category isomorphism

$$(W, w_0, w_2) : {}_{\Bbbk[\langle g\rangle]_q^{a,b}}\mathfrak{M} \to \mathcal{H}_q^{a,b}(\mathfrak{M}).$$

Proof. By Proposition 3.2 we have a category isomorphism $W : {}_{\Bbbk[\langle g\rangle]}\mathfrak{M} \to \mathcal{H}(\mathfrak{M})$. By Theorem 3.3 the monoidal structures on $\mathcal{H}(\mathfrak{M})$ are in a one to one correspondence with those of ${}_{\Bbbk[\langle g\rangle]}\mathfrak{M}$. So, according to Theorem 2.1, the monoidal structures on $\mathcal{H}(\mathfrak{M})$ induced by the strict monoidal structure of $\mathfrak{M}$ are given by the quasi-bialgebra structures of $\Bbbk[\langle g\rangle]$. Using Theorem 2.8 we get that, up to isomorphism, the desired monoidal structures on $\mathcal{H}(\mathfrak{M})$ are completely determined by triples $(q,a,b) \in (\Bbbk\backslash\{0\}) \times \mathbb{Z} \times \mathbb{Z}$ as follows.

Let $(X,f_X), (Y,f_Y), (Z,f_Z)$ be objects in $\mathcal{H}(\mathfrak{M})$. The tensor product in $\mathcal{H}(\mathfrak{M})$ is then given by

$$(X,f_X) \otimes (Y,f_Y) := W\left((X,\mu_X) \otimes (Y,\mu_Y)\right) = W\left(X\otimes Y, \mu_{X\otimes Y}\right) = \left(X \otimes Y, f_{X\otimes Y}\right),$$

where

$$\begin{aligned}
f_{X\otimes Y}(x\otimes y) &= \mu_{X\otimes Y}(g \otimes (x \otimes y)) = g(x\otimes y) = \Delta(g)(x\otimes y)\\
&= gx \otimes gy = f_X(x) \otimes f_Y(y) = (f_X \otimes f_Y)(x \otimes y),
\end{aligned}$$

and so $(X,f_X) \otimes (Y,f_Y) = (X\otimes Y, f_X \otimes f_Y)$. The unit is $W(\Bbbk, \mu_\Bbbk) = (\Bbbk, \mathrm{Id}_\Bbbk)$ since

$$f_\Bbbk(\kappa) := \mu_\Bbbk(g \otimes \kappa) = g \cdot \kappa = \varepsilon(g)\kappa = \kappa,$$

for all $\kappa \in \Bbbk$. The left unit constraint is given, for every $\kappa \in \Bbbk$, $x \in X$, by

$$\begin{aligned}
l_{(X,f_X)}(\kappa \otimes x) &= \left(Wl_{(X,\mu_X)}\right)(\kappa\otimes x) = l_{(X,\mu_X)}(\kappa \otimes x)\\
&= \kappa l_{(X,\mu_X)}(1_\Bbbk \otimes x) = \kappa(\lambda x) = \kappa\left(qg^{-b}x\right) = \kappa q f_X^{-b}(x).
\end{aligned}$$

Likewise, the right unit constraint is given, for every $\kappa \in \Bbbk$, $x \in X$, by

$$\begin{aligned}
r_{(X,f_X)}(x \otimes \kappa) &= \left(Wr_{(X,\mu_X)}\right)(x\otimes \kappa) = r_{(X,\mu_X)}(x \otimes \kappa)\\
&= \left(r_{(X,\mu_X)}\right)(x \otimes 1_\Bbbk)\kappa = (\rho x)\kappa = \kappa\left(qg^{a}x\right) = \kappa q f_X^{a}(x).
\end{aligned}$$

In a similar manner we compute that the associativity constraint is given, for every $x \in X$, $y \in Y$, $z \in Z$, by

$$\begin{aligned}
a_{(X,f_X),(Y,f_Y),(Z,f_Z)}((x\otimes y)\otimes z) &= \left(Wa_{(X,\mu_X),(Y,\mu_Y),(Z,\mu_Z)}\right)((x\otimes y)\otimes z)\\
&= a_{(X,\mu_X),(Y,\mu_Y),(Z,\mu_Z)}((x\otimes y)\otimes z) = \phi^1 x \otimes \left(\phi^2 y \otimes \phi^3 z\right)\\
&= g^a x \otimes \left(y \otimes g^b z\right) = f_X^a(x) \otimes \left(y \otimes f_Z^b(z)\right).
\end{aligned}$$

Thus if we transport through W the monoidal structure of ${}_{\Bbbk[\langle g\rangle]_p^{a,b}}\mathfrak{M}$ what we get on $\mathcal{H}(\mathfrak{M})$ is the monoidal structure of $\mathcal{H}(\mathfrak{M})_p^{a,b}$, as needed.

Since ${}_{\Bbbk[\langle g\rangle]_p^{a,b}}\mathfrak{M}$ has a unique braided (actually symmetric) monoidal structure by the above comments it follows that $\mathcal{H}(\mathfrak{M})_p^{a,b}$ has a unique braided (actually symmetric) monoidal structure, too. It is given by the braiding defined, for every $x \in X$, $y \in Y$, by

$$\begin{aligned} c_{(X,f_X),(Y,f_Y)}(x\otimes y) &= \left(Wc_{(X,\mu_X),(Y,\mu_Y)}\right)(x\otimes y) = c_{(X,\mu_X),(Y,\mu_Y)}(x\otimes y) \\ &= R^2 y\otimes R^1 x = g^{-a-b}y\otimes g^{a+b}x = f_Y^{-a-b}(y)\otimes f_X^{a+b}(x). \end{aligned}$$

The last assertion follows easily from Theorem 3.3. □

Corollary 3.5. *Let $q \in \Bbbk\backslash\{0\}$ and $a, b \in \mathbb{Z}$. We have isomorphisms of symmetric monoidal categories*

$$\mathcal{H}_q^{a,b}(\mathfrak{M}) \cong {}_{\Bbbk\langle g\rangle_q^{a,b}}\mathfrak{M} \cong {}_{\Bbbk[\langle g\rangle]}\mathfrak{M} \cong \mathcal{H}_1^{0,0}(\mathfrak{M}).$$

Proof. By Theorem 3.4 we have $\mathcal{H}_q^{a,b}(\mathfrak{M}) \cong {}_{\Bbbk\langle g\rangle_q^{a,b}}\mathfrak{M}$, and by Corollary 2.11 that ${}_{\Bbbk\langle g\rangle_q^{a,b}}\mathfrak{M} \cong {}_{\Bbbk[\langle g\rangle]}\mathfrak{M}$. Both of them are isomorphisms of symmetric monoidal categories. □

Definition 3.6. Let $(\mathcal{C}, \otimes, \mathbf{1}, a, l, r)$ be a monoidal category. Following [CG, Section 1], the category $\mathcal{H}(\mathcal{C})$ becomes a monoidal category $\mathcal{H}(\mathcal{C}) = (\mathcal{H}(\mathcal{C}), \otimes, (\mathbf{1}, Id_{\mathbf{1}}), a, l, r)$. Here by abuse of notation we denote with the same letters the constraints of $\mathcal{C}$ regarded as morphisms in $\mathcal{H}(\mathcal{C})$ (thus, for instance $a_{(M,f_M)}$ is a_M regarded as a morphism in $\mathcal{H}(\mathcal{C})$). The tensor product of (M, f_M) and (N, f_N) is given by the formula

$$(M, f_M)\otimes(N, f_N) = (M\otimes N, f_M\otimes f_N).$$

At the level of morphisms, the tensor product is the tensor product of morphisms.

In [CG, Proposition 1.1], a modified version $\widetilde{\mathcal{H}}(\mathcal{C}) = \left(\mathcal{H}(\mathcal{C}), \otimes, (\mathbf{1}, Id_{\mathbf{1}}), \widetilde{a}, \widetilde{l}, \widetilde{r}\right)$ of the monoidal category $\mathcal{H}(\mathcal{C})$ was given. Namely, the associativity constraint $\widetilde{a}$ is defined, for (M, f_M), (N, f_N), $(P, f_P) \in \mathcal{H}(\mathcal{C})$, by the formula

$$\widetilde{a}_{(M,f_M),(N,f_N),(P,f_P)} = a_{M,N,P}\circ\left((f_M\otimes N)\otimes f_P^{-1}\right) = \left(f_M\otimes\left(N\otimes f_P^{-1}\right)\right)\circ a_{M,N,P},$$

while the unit constraints $\widetilde{l}$ and $\widetilde{r}$ are defined by

$$\widetilde{l}_{(M,f_M)} = f_M\circ l_M = l_M\circ(\mathbf{1}\otimes f_M) \qquad\text{and}\qquad \widetilde{r}_{(M,f_M)} = f_M\circ r_M = r_M\circ(f_M\otimes\mathbf{1}).$$

Furthermore, by [CG, Proposition 1.2], if $(\mathcal{C}, \otimes, \mathbf{1}, a, l, r, c)$ is a braided monoidal category then so is $\widetilde{\mathcal{H}}(\mathcal{C}) = \left(\mathcal{H}(\mathcal{C}), \otimes, (\mathbf{1}, Id_{\mathbf{1}}), \widetilde{a}, \widetilde{l}, \widetilde{r}, c\right)$.

Hence, by [CG, Proposition 1.7], we deduce that $(\mathcal{H}(\mathcal{C}), \otimes, (\mathbf{1}, Id_{\mathbf{1}}), a, l, r, c)$ is braided as well.

As a consequence of Theorem 3.4, we have an alternative description for the symmetric monoidal categories given in Definition 3.6, providing that $\mathcal{C} = \mathfrak{M}$.

Proposition 3.7. *We have the following equalities of braided monoidal categories*

$$\mathcal{H}(\mathfrak{M}) = \mathcal{H}_1^{0,0}(\mathfrak{M}) \qquad\textit{and}\qquad \widetilde{\mathcal{H}}(\mathfrak{M}) = \mathcal{H}_1^{1,-1}(\mathfrak{M}).$$

The result below gives a more conceptual proof for [CG, Proposition 1.7], in the particular case when $\mathcal{C} = \mathfrak{M}$.

Corollary 3.8. *We have the following isomorphisms of symmetric monoidal categories*

$$\mathcal{H}(\mathfrak{M}) \cong {}_{\Bbbk\langle g\rangle_1^{0,0}}\mathfrak{M} \cong {}_{\Bbbk[\langle g\rangle]}\mathfrak{M} \cong {}_{\Bbbk\langle g\rangle_1^{1,-1}}\mathfrak{M} \cong \widetilde{\mathcal{H}}(\mathfrak{M}).$$

Proof. It follows by Corollary 3.5. See also Proposition 3.7. □

Acknowledgements. We would like to thank J. Gómez-Torrecillas and S. Gelaki for helpful discussions on the main topics of the paper.

References

[BCT] D. Bulacu, S. Caenepeel, B. Torrecillas, *The braided monoidal structures on the category of vector spaces graded by the Klein group*. Proc. Edinb. Math. Soc., II. **54** (2011), 613-641.

[Ca] S. Caenepeel, *Brauer groups, Hopf algebras and Galois theory*. K-Monographs Math. **4**, Kluwer Academic Publishers, Dordrecht, 1998.

[CG] S. Caenepeel, I. Goyvaerts, *Monoidal Hom-Hopf algebras*. Comm. Algebra **39** (2011), no. 6, 2216–2240.

[Dr] V. G. Drinfeld, *Quasi-Hopf algebras*. (Russian) Algebra i Analiz **1** (1989), no. 6, 114–148; translation in Leningrad Math. J. **1** (1990), no. 6, 1419–1457.

[GH] R. Gilmer; R.C. Heitmann, *The group of units of a commutative semigroup ring*. Pacific J. Math. **85** (1979), no. 1, 49–64.

[Ka] C. Kassel, *Quantum groups*. Graduate Texts in Mathematics, **155**. Springer-Verlag, New York, 1995.

[Pa] D. S. Passman, *The algebraic structure of group rings. Pure and Applied Mathematics*. Wiley-Interscience [John Wiley & Sons], New York-London-Sydney, 1977.

[SR] N. Saavedra Rivano, *Catégories Tannakiennes*. Lecture Notes in Mathematics, Vol. **265**. Springer-Verlag, Berlin-New York, 1972.

University of Turin, Department of Mathematics "G. Peano", via Carlo Alberto 10, I-10123 Torino, Italy

E-mail address: alessandro.ardizzoni@unito.it

URL: www.unito.it/persone/alessandro.ardizzoni

Faculty of Mathematics and Informatics, University of Bucharest, Str. Academiei 14, RO-010014 Bucharest 1, Romania

E-mail address: daniel.bulacu@fmi.unibuc.ro

URL: http://fmi.unibuc.ro/ro/departamente/matematica/bulacu_daniel/

University of Ferrara, Department of Mathematics, Via Machiavelli 35, Ferrara, I-44121, Italy

E-mail address: men@unife.it

URL: http://www.unife.it/utenti/claudia.menini